CHAPTER 1 SQUARES AND SQUARE ROOTS

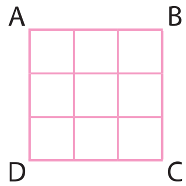
INTRODUCTION

In previous classes, we have already studied the squares of many natural numbers.

For example, $3^2 = 3 \times 3$

We say that 3 to the power 2 or 3 squared is 9.

$$3^2 = 3 \times 3 = 9$$



Now, let us take a square figure ABCD in order to explain the given example. Here, each side of the square has 3 units.

$\therefore$ Area $= 3 \times 3 = 3^2$ square units

$= 9$ square units

In this Chapter, we shall be concentrating on the procedures to find the positive square roots of positive rational numbers.

SQUARES

Look at the examples given below:

$$2 \times 2 = 4 = 2^2$$

$$3 \times 3 = 9 = 3^2$$

$$4 \times 4 = 16 = 4^2$$

Similarly, $a \times a = a^2$

So, we conclude that—

The square of a number is the product obtained by multiplying the number by itself.

Numbers, such as 1, 4, 9, 16, 25, 36 are called **perfect squares**.

Remember

A given number is called a **perfect square** or a **square number** if it is the square of some natural number. These numbers are exact squares and do not involve any decimals or fractions.

To find out whether a given number is a perfect square or not, write the number as a product of its prime factors. If these factors exist in pairs, the number is a perfect square.

Let us take an example to find whether the given number is a perfect square or not.

Example 1: Which of the following numbers are perfect squares?

(i) 256 (ii) 154 (iii) 720

Solution: (i) **Step 1:** $256 = \underline{2 \times 2} \times \underline{2 \times 2} \times \underline{2 \times 2} \times \underline{2 \times 2}$

Step 2: Prime factors of 256 can be grouped into pairs and no factor is left out.

$\Rightarrow 256 = (2 \times 2 \times 2 \times 2)^2 = (16)^2$

$\therefore$ 256 is a perfect square of 16.

(ii) **Step 1:** $154 = 2 \times 7 \times 11$

Step 2: No prime factor exists in pairs.

$\therefore$ 154 is not a perfect square.

(iii) **Step 1:** $720 = \underline{2 \times 2} \times \underline{2 \times 2} \times \underline{3 \times 3} \times 5$

Step 2: In prime factors of 720, factor 5 is left ungrouped.

$\therefore$ 720 is not a perfect square.

Facts About Perfect Squares

(i) A number ending with an odd number of zeroes (one zero, three zeroes and so on) is never a perfect square,

e.g. 150, 25000, 350 are not perfect squares.

(ii) Squares of even numbers are always even,

e.g. $8^2 = 64$ $12^2 = 144$ $20^2 = 400$

(iii) Squares of odd numbers are always odd,

e.g. $7^2 = 49$ $13^2 = 169$ $21^2 = 441$

(iv) The numbers ending with 2, 3, 7, 8 are not perfect squares,

e.g. 32, 243, 37, 368 are not perfect squares.

(v) The square of a number other than 0 and 1, is either a multiple of 3 or exceeds the multiple of 3 by 1.

- Examples of multiples of 3.

  $3^2 = 9$ $\qquad$ $12^2 = 144$

- Examples of multiples of 3 exceeded by 1.

  $4^2 = 16 = (15 + 1)$ $\qquad$ $13^2 = 169 = (168 + 1)$

(vi) The square of a number other than 0 and 1, is either a multiple of 4 or exceeds a multiple of 4 by 1.

- Examples of multiples of 4.

  $6^2 = 36$ $\qquad$ $8^2 = 64$

- Examples of multiples of 4 exceeded by 1.

  $7^2 = 49 = (48 + 1)$ $\qquad$ $9^2 = 81 = (80 + 1)$

(vii) The difference between the squares of two consecutive natural numbers is equal to their sum.

Let us take two consecutive natural numbers, 3 and 4.

$4^2 - 3^2 = 16 - 9 = 7 = 4 + 3$

Thus, in general, if n and $(n + 1)$ be two consecutive natural numbers,

$$\text{then } (n + 1)^2 - n^2 = \left[(n + 1)\,(n + 1)\right] - n^2$$
$$= n^2 + n + n + 1 - n^2 = n + (n + 1)$$

(viii) The square of a natural number n is equal to the sum of the first n odd natural numbers,

e.g. $1^2 = 1$ = sum of the first one odd natural number

$2^2 = 1 + 3$ = sum of the first two odd natural numbers

$3^2 = 1 + 3 + 5$ = sum of the first three odd natural numbers

$4^2 = 1 + 3 + 5 + 7$ = sum of the first four odd natural numbers and so on.

(ix) Squares of natural numbers composed of only digit 1, follow a peculiar pattern.

$1^2 = 1$

$11^2 = 121$

$111^2 = 12321$

$1111^2 = 1234321$

$11111^2 = 123454321$

We can also observe that the sum of the digits of every such number is a perfect square 1, 121, 12321, 1234321.

$$1 = 1 = 1^2$$
$$1 + 2 + 1 = 4 = 2^2$$
$$1 + 2 + 3 + 2 + 1 = 9 = 3^2$$
$$1 + 2 + 3 + 4 + 3 + 2 + 1 = 16 = 4^2$$
$$1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1 = 25 = 5^2$$

See, how beautiful patterns of numbers are made above.

Some Interesting Patterns

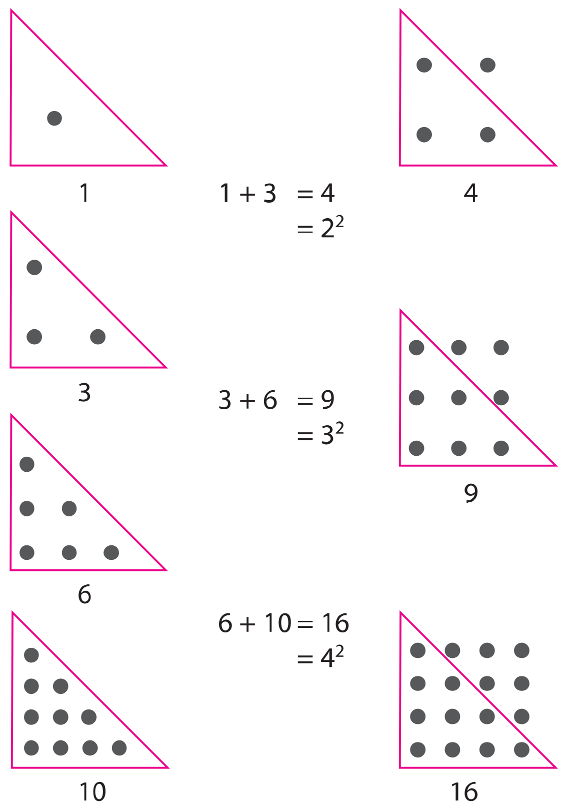
▶ Adding triangular numbers

Remember

Numbers whose dot patterns can be arranged as triangles are called **triangular numbers**.

Let us add triangular numbers.

Triangular Numbers | Combining two consecutive triangular numbers



$1 + 3 = 4 = 2^2$

$3 + 6 = 9 = 3^2$

$6 + 10 = 16 = 4^2$

Observe the following pattern and fill in the blanks.

$$1 + 3 = 2^2$$
$$1 + 3 + 5 = 3^2$$
$$1 + 3 + 5 + 7 = \_\_\_\_\_\_$$
$$1 + 3 + 5 + 7 + 9 = \_\_\_\_\_\_$$
$$1 + 3 + 5 + 7 + 9 + 11 = \_\_\_\_\_\_$$
$$1 + 3 + 5 + 7 + 9 + 11 + 13 = \_\_\_\_\_\_$$

▶ Numbers between square numbers

Let us observe some interesting patterns between two consecutive square numbers.

We have $1^2 = 1$

$2^2 = 4$

The non-square numbers between 1 and 4 are 2, 3.

1, [2, 3,] 4 ⟶ **2 non-square numbers**

The non-square numbers between 4 (= 2^2) and 9 (=3^2) are 5, 6, 7, 8.

4, [5, 6, 7, 8,] 9 ⟶ **4 non-square numbers**

Now, let us put our observations in a tabular form.

Consecutive square numbers	Non-square numbers	Number of non-square numbers
1 and 4	2, 3	2
4 and 9	5, 6, 7, 8	4
9 and 16	10, 11, 12, 13, 14, 15	6
16 and 25	17, 18, ... 24	8
25 and 36	26, 27, ... 35	10

and so on.

Now, let us generalise our observations.

Between 1^2 (= 1) and 2^2 (= 4), there are two non-square numbers (i.e. 2 *x* 1).

Between 2^2 (= 4) and 3^2 (= 9), there are four non-square numbers (i.e. 2 *x* 2).

Between 3^2 (= 9) and 4^2 (= 16), there are six non-square numbers (i.e. 2 *x* 3).

Between 4^2 (= 16) and 5^2 (= 25), there are eight non-square numbers (i.e. 2 *x* 4).

Between 5^2 (= 25) and 6^2 (= 36), there are ten non-square numbers (i.e. 2 *x* 5).

Can you say how many non-square numbers are there between 6^2 and 7^2?

We find that if we take any natural number, n and $(n + 1)$, the number of non-square numbers between n^2 and $(n + 1)^2$ is $2n$.

There are $2n$ non-perfect square numbers between the square of the numbers, n and $(n + 1)$.

Worksheet 1

1. **Which of the following numbers are perfect squares?**

   11, 16, 32, 36, 50, 64, 75

2. **Which of the following numbers are perfect squares of even numbers?**

   121, 225, 784, 841, 576, 6561

3. **Which of the following numbers are perfect squares?**

   100, 205000, 3610000, 212300000

4. **By just observing the digits at ones place, tell which of the following can be perfect squares?**

   1026, 1022, 1024, 1027

5. **How many non-square numbers lie between the following pairs of numbers?**

   (i) 7^2 and 8^2 (ii) 10^2 and 11^2 (iii) 40^2 and 41^2

   (iv) 80^2 and 81^2 (v) 101^2 and 102^2 (vi) 205^2 and 206^2

6. **Write down the correct number in the box.**

   (i) $100^2 - 99^2$ = ☐ = ☐

   (ii) $27^2 - 26^2$ = ☐ = ☐

   (iii) $569^2 - 568^2$ = ☐ = ☐

7. **Observe the pattern in the following and find the missing numbers.**

$$1\underline{2}1 = \frac{(22)^2}{1+2+1}$$

$$12\underline{3}21 = \frac{(333)^2}{1+2+3+2+1}$$

$$123\underline{4}321 = \_\_\_\_\_\_\_\_$$

$$1234\underline{5}4321 = \_\_\_\_\_\_\_\_$$

$$12345\underline{6}54321 = \_\_\_\_\_\_\_\_$$

8. **Which of the following triplets are Pythagorean?**

(3, 4, 5), (6, 7, 8), (10, 24, 26), (2, 3, 4)

[**Hint** : Let the smallest even number be $2m$ and find m from it. Then, find ($2m$, $m^2 - 1$, $m^2 + 1$). If you get the triplet, it is Pythagorean.]

Another way of finding a Pythagorean triplet is:

If '*a*', '*b*' and '*c*' are three natural numbers with '*a*' as the smallest of them, then,

(i) If '*a*' is odd, sum of other two numbers is a^2 and their difference is 1.

(ii) If '*a*' is even, sum of other two numbers is $\frac{a^2}{2}$ and their difference is 2.

SQUARE ROOTS

We know that

$$4^2 = 4 \times 4 = 16$$

We say square root of 16 is 4.

This is written as $\sqrt{16} = 4$.

> **Note:** $(-4)^2 = 16$ Therefore, square root of 16 can be –4 also, but here we are taking only positive square root.

Let us see some more examples.

$$7^2 = 49 \longrightarrow \sqrt{49} = 7$$

$$5^2 = 25 \longrightarrow \sqrt{25} = 5$$

$$8^2 = 64 \longrightarrow \sqrt{64} = 8$$

In general, if $m^2 = n$ then $\sqrt{n} = m$

Hence, square root of a given natural number n is that natural number m whose square is n.

From the above examples, we observe that—

(i) the square root of an even number is even.

(ii) the square root of an odd number is odd.

(iii) the symbol for the square root is $\sqrt{\ }$.

(iv) if a is the square root of b, then b is the square of a.

Observe the following pattern.

$$1 + 3 = 2^2$$
$$1 + 3 + 5 = 3^2$$
$$1 + 3 + 5 + 7 = 4^2$$
$$1 + 3 + 5 + 7 + 9 + 11 + 13 = 7^2$$
$$1 + 3 + 5 + 7 + \ldots\ldots\ldots\ldots \text{ up to } n \text{ terms} = n^2$$

The sum of first n odd numbers is n^2.

Finding Square Root of a Number by the Repeated Subtraction Method

Let us find $\sqrt{9}$.

Step 1: $9 - 1 = 8$ → First odd number

Step 2: $8 - 3 = 5$ → Second odd number

Step 3: $5 - 5 = 0$ → Third odd number

We have subtracted from 9, the successive odd numbers 1, 3 and 5, and obtained 0 in **Step 3**.

$\therefore \sqrt{9} = 3$

Consider another example.

Example 2: Find $\sqrt{121}$ by repeated subtraction method.

Solution : **Step 1:** $121 - 1 = 120$ **Step 3:** $117 - 5 = 112$

Step 2: $120 - 3 = 117$ **Step 4:** $112 - 7 = 105$

Step 5: 105 – 9 = 96

Step 6: 96 – 11 = 85

Step 7: 85 – 13 = 72

Step 8: 72 – 15 = 57

Step 9: 57 – 17 = 40

Step 10: 40 – 19 = 21

Step 11: 21 – 21 = 0

We have subtracted from 121, the successive odd numbers from 1 to 21, and obtained 0 in **Step 11**.

$\therefore \quad \sqrt{121} = 11$

Worksheet 2

Find the square root of the following numbers by the repeated subtraction method.

(i)	16	(ii)	49	(iii)	64	(iv)	100
(v)	169	(vi)	81	(vii)	256	(viii)	144

Finding Square Root by Prime Factorisation

To find the square root of a perfect square by prime factorisation, we go through the following steps:

I. Write down the prime factors of the given number.

II. Make pairs of prime factors such that both the factors in each pair are equal.

III. Write one factor from each pair.

IV. Find the product of the above factors.

V. The product is the required square root.

Let us now take some examples to find the square root by prime factorisation.

Example 3: Find the square root of 1156.

Solution:

$$1156 = \underline{2 \times 2} \times \underline{17 \times 17}$$

$$\sqrt{1156} = \sqrt{\underline{2\times2}\times\underline{17\times17}}$$

$$\sqrt{1156} = 2 \times 17$$

$$= 34$$

2	1156
2	578
17	289
17	17
	1

Therefore, the square root of 1156 is 34.

Example 4: Find the square root of 11025.

3	11025
3	3675
5	1225
5	245
7	49
7	7
	1

Solution:

$$11025 = \underline{3 \times 3} \times \underline{5 \times 5} \times \underline{7 \times 7}$$

$$\sqrt{11025} = \sqrt{\underline{3\times3}\times\underline{5\times5}\times\underline{7\times7}}$$

$$\sqrt{11025} = 3 \times 5 \times 7$$

$$= 105$$

Therefore, the square root of 11025 is 105.

Example 5: Find the smallest number by which 9408 must be divided so that it becomes a perfect square. Also, find the square root of the number so obtained.

2	9408
2	4704
2	2352
2	1176
2	588
2	294
7	147
7	21
3	3
	1

Solution:

$$9408 = \underline{2 \times 2} \times \underline{2 \times 2} \times \underline{2 \times 2} \times \underline{7 \times 7} \times 3$$

We observe that prime factor 3 does not form a pair.

Therefore, we must divide the number by 3 so that the quotient becomes a perfect square.

$$\therefore \quad \frac{9408}{3} = 3136$$

$$3136 = (2 \times 2) \times (2 \times 2) \times (2 \times 2) \times (7 \times 7)$$

Now, each prime factor occurs in pairs. Therefore, the required smallest number is 3.

$$\therefore \quad \sqrt{3136} = 2 \times 2 \times 2 \times 7 = 56$$

Worksheet 3

1. **Find the square root of each of the following by prime factorisation.**

   (i) 225   (ii) 441   (iii) 529   (iv) 40000

   (v) 7744   (vi) 8281   (vii) 4096   (viii) 28900

2. **Find the smallest number by which 1100 must be multiplied so that the product becomes a perfect square. Also, find the square root of the perfect square so obtained.**

3. **By what smallest number must 180 be multiplied so that it becomes a perfect square? Also, find the square root of the number so obtained.**